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Orbifold compactification and solutions of M-theory from Milne spaces

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Abstract. In this paper, we consider solutions and spectral functions of M-theory from Milne spaces with extra free dimensions. Conformal deformations to the metric associated with real hyperbolic space forms are derived. For the three-dimensional case, the orbifold identifications $\mathrm{SL}(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \mathrm{Id}\}$, where Id is the identity matrix, is analyzed in detail. The spectrum of an eleven-dimensional field theory can be obtained with the help of the theory of harmonic functions in the fundamental domain of this group and it is associated with the cusp forms and the Eisenstein series. The supersymmetry surviving for supergravity solutions involving real hyperbolic space factors is briefly discussed.

1 Introduction

The true nature of gravity, though well measured by solar system scale experiments, is not well determined at larger or smaller scales. For example, the assumptions of dark matter and/or energy are made in order to fit the observed Universe within Einstein theory, yet it is quite possible that it is the theory of gravity which should be modified to accomodate these data. Theoretically, the possibility that gravity might not be fundamentally described by a purely tensorial theory in four dimensions is growing in importance. This is in part a consequence of superstring theory, which is consistent in ten dimensions (or M-theory in eleven dimensions), but also the more phenomenological recent developments of "braneworld" cosmological scenarios [1–5] have motivated the study of other gravitational theories in four dimensions.

It is generally accepted that M-theory may provide a consistent quantum theory of gravity. Nevertheless, it is understood by now that to insert this theory in time-dependent backgrounds can bring about a number of technical problems such as the appearance of closed timelike curves and the spacetime resulting from string compactification not being Hausdorff. Yet, open questions in cosmology such as the initial (big bang) singularity and the initial boundary conditions remain a challenge and they are the main motivations to consider string cosmology.

Initial boundary conditions and the requirement of homogeneity for the cosmological solution imply that the geometry has the form of a higher-dimensional Milne universe along a null hypersurface, with negative constant curvature in the spatial sector. This spacetime can be viewed particularly as hyperbolic compactifications in M-theory [6–8], which have recently attracted some interest as they lead to interesting cosmologies [9]. Cosmological string models in a Milne universe have been considered by many authors. Milne spaces in the context of inflationary cosmology were studied in [10, 11]. String models in $(1 + 1)$-dimensional Milne space were discussed in [6, 7, 12–16]. Discussions of higher-dimensional Milne spaces can be found in [6, 12], and in the more recent papers in [7, 15].

In the present paper we will extend these previous works in order to contemplate the problem of hyperbolic compactifications in the context of cosmological scenarios. In particular, we will be interested in the general class of time-dependent locally flat spacetimes obtained from the $(N+1)$-dimensional Milne universe. Emphasis is put to the $N = 3$ case, which is analyzed in detail, and orbifold identifications using the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \mathrm{Id}\}$, where Id is the identity matrix is considered. In Sect. 2 we analyze the class of conformal deformations of Riemannian metrics and in particular the conformal relation between Milne and hyperbolic space forms. In Sect. 3 we take into account the constant slice in Milne spaces. In Sect. 4 we consider hyperbolic geometry in the spatial section of the Milne space. For co-compact groups Γ (i.e. for compact real hyperbolic manifolds) the heat-kernel coefficients are given in explicit forms. Orbifolding of the group Γ we derived in explicit forms the Selberg trace formula and the determinant of Laplace type operators. The study of this theory involves harmonic analysis on locally symmetric spaces of rank one from which we will extract some results for the brane picture. Finally in Sect. 5 we discuss questions of supersymmetry surviving under the orbifolding of a discrete group.

2 Conformal deformations and Milne space forms

Let M be a $D=(N+1)$-dimensional Riemannian space with metric

$$\mathrm{d}s^2 = g_{00}(\mathbf{x})(\mathrm{d}x^0)^2 + g_{ij}(\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j\,, \mathbf{x}=\{x^j\}\,,\quad i,j=1,\ldots,N.$$

For the conformal deformations of the $g_{\mu\nu}$ the following relation holds:

$$\widetilde{g}_{\mu\nu}(\mathbf{x}) = \mathrm{e}^{2\sigma(\mathbf{x})} g_{\mu\nu}(\mathbf{x})\,,\quad \sigma(\mathbf{x})\in C^\infty(M). \tag{2.1}$$

Recalling that the partition function of field theory is given by (in the Euclidean sector differential operators are elliptic) $W=\int \mathrm{d}[\varphi]\,\exp\left(-(1/2)\int_M \mathrm{d}^D x\varphi\mathfrak{L}\varphi\right)$, where φ is a scalar density of weight $-1/2$ and the operator $\mathfrak{L}$ has the form $\mathfrak{L}=-\triangle^g+m^2+\xi R^g$, where m (the mass) and ξ are arbitrary parameters, while $\triangle^g$ and R^g are respectively the Laplace–Beltrami operator and the scalar curvature of the space with respect to the original metric g. One gets the following result, a proposition due to Bytsenko, Cognola and Zerbini [17].

Let $\widetilde{\varphi}=\mathrm{e}^{\sigma}\varphi$; then the conformal deformations (2.1) lead to $\widetilde{\mathfrak{L}}=\mathrm{e}^{-\sigma}\mathfrak{L}\mathrm{e}^{\sigma}$, and

$$\begin{aligned}
R^{\widetilde{g}} &= \mathrm{e}^{-2\sigma}\left[R^g - 2(D-1)\triangle^g\sigma - (D-1)(D-2)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]\,,\\
\triangle^{\widetilde{g}}\widetilde{\varphi} &= \frac{1}{4}\mathrm{e}^{-\sigma} \times\left[4\triangle^g - 2(D-2)\triangle^g\sigma - (D-2)^2 g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]\varphi\\
&= \mathrm{e}^{-\sigma}\left[\triangle^g + \xi_D(\mathrm{e}^{2\sigma}R^{\widetilde{g}} - R^g)\right]\varphi\,, \qquad (2.2)\\
\mathfrak{L} &= \mathrm{e}^{\sigma}\left\{-\triangle^{\widetilde{g}} + \xi R^{\widetilde{g}} + \mathrm{e}^{-2\sigma}\left[m^2 + (\xi-\xi_D)R^g\right]\right\}\mathrm{e}^{\sigma},
\end{aligned}$$

where $\xi_D=(D-2)/4(D-1)$ is the conformal invariant factor.

The classical conformal invariance requires that the action S is invariant in form, that is $\widetilde{S}=S[\widetilde{\varphi},\widetilde{g}]$ (which is to say $\widetilde{\mathfrak{L}}=\mathfrak{L}$). As it is well known, this happens only for conformally coupled massless fields ($\xi=\xi_D$). For the partition function we have $\widetilde{W}=J[g,\widetilde{g}]\,W$, where $J[g,\widetilde{g}]$ is the Jacobian of the conformal deformation.

For $0<t<1$ the asymptotic expansion holds

$$\mathrm{Tr}\,\mathrm{e}^{-t\mathfrak{L}} \simeq \sum_j A_j(\mathfrak{L})t^{(j-D)/2}\,,$$

$$A_j(\mathfrak{L}) = (4\pi)^{-D/2}\int_M \mathrm{d}^D x\sqrt{g}a_j(x|\mathfrak{L}), \tag{2.3}$$

where $a_j(x|\mathfrak{L})$ is the jth Seeley–De Witt coefficient (in conformal invariant theories it is proportional to the trace anomaly). If the boundary of a manifold is empty then $A_j(\mathfrak{L})=0$ for any odd j. The results of the following proposition hold.

Let us consider a family of conformal deformations

$$g^q_{\mu\nu} = \mathrm{e}^{2q\sigma}g_{\mu\nu} = \mathrm{e}^{2(q-1)\sigma}\widetilde{g}_{\mu\nu}\,,$$

$$\sqrt{g^q} \equiv \sqrt{|\det g^q_{\mu\nu}|} = \mathrm{e}^{Dq\sigma}\sqrt{g}\,.$$

The metric is $g_{\mu\nu}$ or $\widetilde{g}_{\mu\nu}$ according to whether $q=0$ or $q=1$ respectively. Then

$$\begin{aligned}
\log J[g_q, g_{q+\delta q}] &= \log\left[W_{q+\delta q}/W_q\right]\\
&= (4\pi)^{-D/2}\delta q\int_M \mathrm{d}^D x\sqrt{g^q}a_D(x|\mathfrak{L}^q)\sigma(x)\,, \qquad (2.4)\\
\log J[g,\widetilde{g}] &= (4\pi)^{-D/2}\int_0^1 \mathrm{d}q\int_M \mathrm{d}^D x\sqrt{g^q}\,a_D(x|\mathfrak{L}^q)\sigma(x)\,, \qquad (2.5)\\
\log W &= \log\widetilde{W} - \log J[g,\widetilde{g}]\\
&= \frac{\mathrm{d}}{2\mathrm{d}s}\zeta(s|\widetilde{\mathfrak{L}}\ell^2)|_{s=0} - \log J[g,\widetilde{g}]\,. \qquad (2.6)
\end{aligned}$$

Equation (2.6) has been derived with the help of the zeta-function regularization, ℓ being an arbitrary parameter necessary to adjust the dimensions.

2.1 Remark

The $(N+1)$-dimensional Milne space is described by the metric $\mathrm{d}s^2=-\mathrm{d}t^2+t^2\mathrm{d}\mathbb{H}^N$, where $\mathrm{d}\mathbb{H}^N$ is the arc element of the hyperboloid or upper half N-plane. The space is flat, as it is evident upon introducing Cartesian coordinates as follows: $U=ty^{-1}$, $V=ty+U\sum_{j=1}^{N-1}x_j^2$, $X_j=Ux_j$. This provides the embedding of the hyperboloid in $(N+1)$-Minkowski space, $\mathrm{d}s^2=-\mathrm{d}U\mathrm{d}V+\mathrm{d}X_j^2$, where the hyperboloid is described by $t^2=UV-\sum_{j=1}^{N-1}X_j^2$, which exhibits the $\mathrm{SO}_1(N,1)$ isometry of $\mathbb{H}^N$.

Before concluding this section, some remarks on the Milne metric are in order. New coordinates in the Euclidean sector, $t\to it$, can be introduced as follows: $\tau=\log t, t\neq 0$ ($t=0$ is a harmless coordinate singularity and corresponds to a horizon in this metric). This gives the new form for the metric: $\mathrm{d}s^2=\mathrm{e}^{2\tau}\mathrm{d}\tau^2+\mathrm{e}^{2\tau}\mathrm{d}\mathbb{H}^N$. Taking into account (2.1) one can choose $\sigma(\mathbf{x})=-\tau$. In the Euclidean sector it gives

$$\mathrm{d}\widetilde{s}^2 = \mathrm{d}\tau^2 + \mathrm{d}\mathbb{H}^N. \tag{2.7}$$

Therefore, in a class of conformal deformations the metric (2.7) is related to the initial metric of the Milne space and can be associated with spacetime forms of topology $S^1\times\mathbb{H}^N$. One can use angular coordinates and define the initial metric as follows:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \frac{t^2}{\rho^2}(\mathrm{d}\rho^2 + \mathrm{d}\Omega^2_{N-1})\,, \tag{2.8}$$

where $\mathrm{d}\Omega^2_{N-1}$ is the metric of a $(N-1)$-dimensional space. The technique of the conformal deformations of the Rindler space (except for a horizon) with its connection to a space with hyperbolic spatial section has been discussed in [17]. The metrics of both spaces have the form

$$\mathrm{d}s^2_{(\mathrm{Rindler})} = \rho^2\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}\Omega^2_{N-1} \overset{\sigma=-\log\rho}{\Longrightarrow}$$

$$d\widetilde{s}^2_{(S^1\times\mathbb{H}^N)} = dt^2 + \frac{1}{\rho^2}(d\rho^2 + d\Omega^2_{N-1}). \tag{2.9}$$

For the Milne space a similar deformation (except for a horizon) in coordinate τ becomes

$$ds^2_{(\mathrm{Milne})} = e^{2\tau}\left(d\tau^2 + \frac{1}{\rho^2}(d\rho^2 + d\Omega^2_{N-1})\right) \stackrel{\sigma\equiv-\tau}{\Longrightarrow}$$
$$d\widetilde{s}^2_{(S^1\times\mathbb{H}^N)} = d\tau^2 + \frac{1}{\rho^2}(d\rho^2 + d\Omega^2_{N-1}). \tag{2.10}$$

The metrics on Rindler and Milne spaces are in the conformal class, and the connections between their conformal deformations read $\tau = \log\rho$. Here we derive the operator $\widetilde{L}_N \equiv \widetilde{\mathfrak{L}} - \partial^2_\tau$, acting on scalars in the spatial section of the manifold defined by the metric (2.7),

$$\widetilde{L}_N = -\triangle^{\widetilde{g}}_N - \rho^2_N + e^{2\tau}(m^2 + \xi R^g)\,, \tag{2.11}$$
$$\triangle^{\widetilde{g}}_N = \partial^2_\tau - (N-1)\partial_\tau + e^{2\tau}\triangle_{N-1}\,,$$

where $\triangle_{N-1}$ is the Laplace–Beltrami operator on $(N-1)$-dimensional space, $\rho_N = (N-1)/2$. The appearance should be noted of an effective "tachyonic" mass $-\rho^2_N$, which has important consequences for the structure of the zeta-function related to the operator $\widetilde{L}_N$, which has generally speaking a continuum spectrum (see [17] for details). We have the following proposition (Bytsenko, Cognola and Zerbini [17]).

The trace of the heat kernel has the form

$$\mathrm{Tr}\, e^{-t\widetilde{L}_N} = \sum_{n=0}^{(N-3)/2} \frac{A_{2n}(L_{N-1})\, a(t|-\triangle_{\mathbb{H}^{N-2n}} - \rho^2_{N-2n})}{N-1-2n} \times(4\pi\varepsilon^{-2})^{(N-1-2n)/2}$$
$$+\frac{1}{4\sqrt{\pi t}}\left[\frac{d}{ds}\zeta(s|L_{N-1})|_{s=0} - 2\zeta(0|L_{N-1})\log(\varepsilon/2)\right]$$
$$-\frac{1}{4}\zeta(0|L_{N-1}) + \frac{1}{2\pi}\zeta(0|L_{N-1})\int_{\mathbb{R}} dr\psi(ir)e^{-tr^2}, \tag{2.12}$$

$$\mathrm{Tr}\, e^{-t\widetilde{L}_N} = \sum_{n=0}^{(N-2)/2} \frac{A_{2n}(L_{N-1})\, a(t|-\triangle_{\mathbb{H}^{N-2n}} - \rho^2_{N-2n})}{N-1-2n} \times(4\pi\varepsilon^{-2})^{(N-1-2n)/2}$$
$$+\frac{1}{4\sqrt{\pi t}}\frac{d}{ds}\zeta(s|L_{N-1})|_{s=0} \tag{2.13}$$

valid for odd and even N respectively. Here by $a(t|-\triangle_{\mathbb{H}^{N-2n}} - \rho^2_{N-2n})$ we indicate the diagonal heat kernel of a Laplace-like operator on $\mathbb{H}^{N-2n}$, and ε is a horizon cutoff parameter in integrating over coordinates.

3 Constant time slices in Milne cosmology

Now let us consider the eleven-dimensional metric

$$ds^2_{10} = -dt^2 + t^2 d\mathbb{H}^N + dx_1^2 + dx_2^2 + dx_3^2 + \sum_{j=1}^{7-N} dy_j^2\ , \tag{3.1}$$

where the y coordinates describe compact internal dimensions. This is an exact solution of M-theory [6–8]. The internal space described by the y coordinates can be replaced by any Ricci flat space, giving a more general class of cosmological backgrounds. Note that four-dimensional Friedmann–Robertson–Walker cosmology can be obtained from this model [7]. First, we replace the hyperboloid $\mathbb{H}^N$ by a finite volume space $\Gamma\backslash\mathbb{H}^N$, where Γ is a discrete subgroup of isometries such that the space has finite volume. Then we compactify to four dimensions. To obtain the four-dimensional Einstein frame metric, we write the metric in the form

$$ds^2 = e^{2a(t)}ds^2_{4E} + e^{2b(t)}d\mathbb{H}^N + \sum_{j=1}^{7-N} dy_j^2\ , \tag{3.2}$$
$$ds^2_{4E} = e^{2c(t)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\ . \tag{3.3}$$

If the condition $e^{2a}e^{Nb} = 1$ is satisfied, then ds^2_{4E} is the Einstein frame metric. Comparing to (3.1), one obtains

$$ds^2_{4E} = t^N(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\ , \tag{3.4}$$

or

$$ds^2_{4E} = -d\tau^2 + \tau^{2N/(N+2)}(dx_1^2 + dx_2^2 + dx_3^2). \tag{3.5}$$

This corresponds to 4D Einstein equations coupled to an energy momentum tensor of a perfect fluid with equation of state $p = \kappa\rho$, $\kappa = (4-D)/3D$. Although we have started with vacuum Einstein equations in ten dimensions, the four-dimensional Einstein metric describes a homogeneous and isotropic space in the presence of matter. This matter is, of course, the scalar field associated with the modulus representing the volume of the hyperbolic space. Interestingly, the above metric is the asymptotic (large time) form of the models of [9]. For $D = 4$, it describes a universe filled with dust ($p = 0$), and for $D > 4$ a universe filled with negative pressure matter[1].

Since the models are based on a flat eleven-dimensional geometry, the $(N+1)$-dimensional Milne universes provide a simple setup for the study of interesting cosmological models. Let us consider strings/branes propagating in this space. An important question is whether the model is exactly solvable. To start with, consider the model based on $\mathbb{H}^N$ with no identification, i.e. Γ is trivial. From the relation $t^2 - UV + \sum_{j=1}^{N-1} X_j^2 = 0$ it follows that $UV - X_j^2 \geq 0$. If the physical space is restricted only to this Milne patch, say with $t > 0$, then the brane coordinates are subject to the constraint that the brane lives in the interior of the future directed light cone; the space is not geodesically complete and a full description requires boundary conditions at the light cone surface. If it is possible that consistency also requires the inclusion of the past light cone (in string theory this is possible), then the geometry would describe a

[1] In passing, we mention that, in Einstein gravity, the accelerated expansion of a spatially flat universe requires the cosmological dynamics to be dominated by some exotic matter with negative pressure. We plan to address this problem in a forthcoming paper.

universe contracting to a big crunch which makes a transition to an expanding big bang universe. It is non-trivial to impose the condition $UV - X_j^2 \geq 0$ in brane theory. On the other hand, if the full space U, V, X_j is considered, closed timelike curves can arise in the exterior of the light cone as a result of the identifications.

4 Hyperbolic geometry in M-theory

Let us consider an irreducible rank one symmetric space $X = G/K$ of non-compact type. Thus G will be a connected non-compact simple split rank one Lie group with finite center and $K \subset G$ will be a maximal compact subgroup. Up to local isomorphism we can represent X by the following quotients:

$$X = \mathrm{SO}_1(N,1)/\mathrm{SO}(N)\,,\ \mathrm{SU}(N,1)/\mathrm{U}(N)\,, \quad \mathrm{SP}(N,1)/(\mathrm{SP}(N)\times\mathrm{SP}(1))\,,\ \mathrm{F}_{4(-20)}/\mathrm{Spin}(9)\,, \tag{4.1}$$

where the dimension of the spaces is $N, 2N, 4N, 16$ respectively, in these cases. For details on these matters the reader may consult [18]. The spherical harmonic analysis on X is controlled by Harish-Chandra's Plancherel density $\mu(r)$, a function on the real numbers $\mathbb{R}$, computed by Miatello [19–21], and others, in the rank one case we are considering. The object of interest is the groups $G = \mathrm{SO}_1(N,1)$ ($N \in \mathbb{Z}_+$) and $K = \mathrm{SO}(N)$. The corresponding symmetric space of non-compact type is the real hyperbolic space $X = \mathbb{H}^N = \mathrm{SO}_1(N,1)/\mathrm{SO}(N)$ of sectional curvature -1. Its compact dual space is the unit N-sphere.

4.1 Co-compact group

Let τ be an irreducible representation of K on a complex vector space V_τ, and form the induced homogeneous vector bundle $G \times_K V_\tau$. Restricting the G action to Γ we obtain the quotient bundle $E_\tau = \Gamma\backslash(G \times_K V_\tau) \longrightarrow X_\Gamma = \Gamma\backslash X$ over X. The natural Riemannian structure on X (therefore on X_Γ) induced by the Killing form (,) of G gives rise to a connection Laplacian L_Γ on E_τ. If Ω_K denotes the Casimir operator of K, that is $\Omega_K = -\sum y_j^2$, for a basis $\{y_j\}$ of the Lie algebra $\mathfrak{k}_0$ of K, where $(y_j\ , y_\ell) = -\delta_{j\ell}$, then $\tau(\Omega_K) = \lambda_\tau \mathbf{1}$ for a suitable scalar λ_τ. Moreover for the Casimir operator Ω of G, with Ω operating on smooth sections $\Gamma^\infty E_\tau$ of E_τ one has $L_\Gamma = \Omega - \lambda_\tau \mathbf{1}$. For $\lambda \geq 0$ let $\Gamma^\infty\,(X_\Gamma\ , E_\tau)_\lambda = \{s \in \Gamma^\infty E_\tau\,|{-L_\Gamma s} = \lambda s\}$ be the space of eigensections of L_Γ corresponding to λ. Here we note that if X_Γ is compact we can order the spectrum of $-L_\Gamma$ by taking $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$; $\lim_{j\to\infty} \lambda_j = \infty$. Then we have the theorem of Bytsenko and Williams [22], which is as follows.

The heat kernel admits an asymptotic expansion (2.3), and for all G except $G = \mathrm{SO}_1(\ell,1)$ with ℓ odd, and for $0 \leq k \leq N/2 - 1$,

$$A_k(L_\Gamma) = (4\pi)^{\frac{N}{2}-1}\chi(1)\,\mathrm{Vol}(\Gamma\backslash G)C_G\pi \times \sum_{\ell=0}^{k} \frac{(-\rho_N^2)^{k-\ell}}{(k-\ell)!}\left[\frac{N}{2} - (\ell+1)\right]!\,a_{2\left[\frac{N}{2}-(\ell+1)\right]}, \tag{4.2}$$

while for $n = 0, 1, 2, \ldots$ we have

$$A_{\frac{N}{2}+n}(L_\Gamma) = (-1)^n(4\pi)^{\frac{N}{2}-1}\chi(1)\,\mathrm{Vol}(\Gamma\backslash G)C_G\pi \times \left[\sum_{j=0}^{\frac{N}{2}-1}(-1)^{j+1}\frac{\rho_N^{2(n+1+j)}j!a_{2j}}{(n+1+j)!} + 2\sum_{j=0}^{\frac{N}{2}-1}\sum_{\ell=0}^{n}(-1)^\ell\frac{\rho_0^{2(n-\ell)}}{(n-\ell)!}\beta_{\ell+1}(j)a_{2j}\right]. \tag{4.3}$$

Here $\beta_r(j)$ ($r \in \mathbb{Z}_+$) is given by

$$\beta_r(j) \stackrel{\mathrm{def}}{=} \left[2^{1-2(r+j)} - 1\right]\left[\frac{\pi}{a(G)}\right]^{2(r+j)} \times \frac{(-1)^j B_{2(r+j)}}{2(r+j)[(r-1)!]}; \tag{4.4}$$

B_r is the rth Bernoulli number, $a(G) \stackrel{\mathrm{def}}{=} \pi$ if $G = \mathrm{SO}_1(\ell,1)$ with ℓ even, and a_{2j}, C_G are some constants (C_G depending on G). For $G = \mathrm{SO}_1(2n+1,1)$, $k = 0, 1, 2, \ldots$

$$A_k(L_\Gamma) = \pi(4\pi)^{n-\frac{1}{2}}\chi(1)\,\mathrm{Vol}(\Gamma\backslash G)C_G \times \sum_{\ell=0}^{\min(k,n)} \frac{(-n^2)^{k-\ell}\Gamma\left(n-\ell+\frac{1}{2}\right)a_{2(n-\ell)}}{(k-\ell)!}. \tag{4.5}$$

4.2 The orbifold coset: $\Gamma = \mathbf{SL}(2, \mathbb{Z} + \mathrm{i}\mathbb{Z})/\{\pm\mathbf{Id}\}$

In [7] the $\mathrm{SL}(2,\mathbb{Z})$ orbifold model from Milne spaces and the string spectrum associated with that orbifold has been analyzed. It has been also shown that strings with $\mathrm{SL}(2,\mathbb{Z})$ identifications are related to the null orbifold [12] with an extra reflection generator.

Here we consider the case $N = 3$ and the group of local isometry associated with a simple three-dimensional complex Lie group. The discrete group can be chosen in the form $\Gamma \subset \mathrm{PSL}(2,\mathbb{C}) \equiv \mathrm{SL}(2,\mathbb{C})/\{\pm\mathrm{Id}\}$, where Id is the 2×2 identity matrix and is an isolated element of Γ. The group Γ acts discontinuously at the point $z \in \bar{\mathbb{C}}$, $\bar{\mathbb{C}}$ being the extended complex plane. We consider a special discrete group $\mathrm{SL}(2,\mathbb{Z}+\mathrm{i}\mathbb{Z})/\{\pm\mathrm{Id}\}$, where $\mathbb{Z}$ is the ring of integer numbers. The element $\gamma \in \Gamma$ will be identified with $-\gamma$. The group Γ has, within a conjugation, one maximal parabolic subgroup Γ_∞. Let us consider an arbitrary integral operator with kernel $k(z,z')$. Invariance of the operator is equivalent to fulfillment of the condition $k(\gamma z, \gamma z') = k(z,z')$ for any $z, z' \in \mathbb{H}^3$ and $\gamma \in \mathrm{PSL}(2,\mathbb{C})$. So the kernel of the invariant operator is a function of the geodesic distance between z and z'. It is convenient to replace such a distance with the fundamental invariant of a pair of points $u(z,z') = |z-z'|^2/yy'$, thus $k(z,z') = k(u(z,z'))$. Let λ_j be the isolated eigenvalues of the self-adjoint extension of the Laplace operator and let us introduce a suitable analytic function $h(r)$ and $r_j^2 = \lambda_j - 1$. It can be shown that $h(r)$ is related to the quantity $k(u(z,\gamma z))$ by means of the Selberg transform. Let us denote by $g(u)$ the Fourier transform of

$h(r)$, namely $g(u) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{d}r\, h(r)\exp(-\mathrm{i}ru)$. We now have the following theorem.

Suppose $h(r)$ be an even analytic function in the strip $|\Im r| < 1+\varepsilon$ ($\varepsilon > 0$), and $h(r) = \mathcal{O}(1+|r|^2)^{-2}$. For the special discrete group $\mathrm{SL}(2,\mathbb{Z}+\mathrm{i}\mathbb{Z})/\{\pm\mathrm{Id}\}$ the Selberg trace formula holds

$$\begin{aligned}\sum_j h(r_j) - &\sum_{\substack{\{\gamma\}_\Gamma,\gamma\neq\mathrm{Id},\\ \gamma\text{-non-parabolic}}} \int \mathrm{d}\mu(z)\, k(u(z,\gamma z)) \\ &-\frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{d}r\, h(r)\frac{\mathrm{d}}{\mathrm{d}s}\log S(s)\big|_{s=1+\mathrm{i}r} \\ &+\frac{h(0)}{4}[S(1)-1] - \mathfrak{C}g(0) \\ = \mathrm{Vol}(\Gamma\backslash G)&\int_0^\infty \frac{\mathrm{d}r\, r^2}{2\pi^2}\, h(r) \\ &-\frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{d}r h(r)\psi(1+\mathrm{i}r/2). \end{aligned} \tag{4.6}$$

The first term in the right hand side of (4.6) is the contribution of the identity element, $\mathrm{Vol}(\Gamma\backslash G)$ is the (finite) volume of the fundamental domain with respect to the measure $\mathrm{d}\mu$, $\psi(s)$ is the logarithmic derivative of the Euler Γ-function, and $\mathfrak{C}$ is a computable real constant [23–25]. The function $S(s)$ is given by a generalized Dirichlet series $S(s) = \pi^{1/2}\Gamma(s-1/2)[\Gamma(s)]^{-1}\sum_{c\neq 0}\sum_{0\le d<|c|}|c|^{-2s}$, where the sums are taken over all pairs c, d of the matrix

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \subset \Gamma_\infty\backslash\Gamma/\Gamma_\infty .$$

The meromorphic function $S(s)$ is convergent for $\Re\, s > 1$, and it poles are contained in the region $\Re\, s < 1/2$ and in the interval $[1/2, 1]$.

In general, the determinant of an elliptic differential operator requires a regularization. It is convenient to introduce the operator $L_\Gamma(\delta) = L_\Gamma + \delta^2 - 1$, with δ a suitable parameter. One of the most widely used regularizations is the zeta-function regularization. Thus, one has $\mathrm{logdet}\, L_\Gamma(\delta) = -(\mathrm{d}/\mathrm{d}s)\zeta(s|L_\Gamma(\delta))|_{(s=0)}$. In standard cases, the zeta-function at $s=0$ is well defined and one gets a finite result. The meromorphic structure of the analytically continued zeta-function is related to the asymptotic properties of the heat-kernel trace. Summarizing, the final result is the theorem due to Bytsenko, Cognola and Zerbini [24].

The following identity holds:

$$\begin{aligned}\det L_\Gamma(\delta) = &\frac{2}{(\pi\delta)^{1/2}\Gamma(\delta/2)} \\ &\times \exp\left(-\frac{1}{6\pi}\mathrm{Vol}(\Gamma\backslash G)\delta^3 + \mathfrak{C}\delta\right) Z_\Gamma(1+\delta),\end{aligned} \tag{4.7}$$

where $Z_\Gamma(s)$ is Selberg's zeta-function.

Let us analyze a scalar field propagating in these orbifolds. Normalizable wave functions associated with a scalar density can be written in terms of cusp forms. Cusp forms are authomorphic functions which decrease exponentially at infinity. The discrete part of the spectrum is associated with cusp forms, while the Eisenstein series is related to the continuous part. A vertex operator of a brane model contains cusp forms. In the string case a computation of S-matrix elements by using plane-wave vertex operators has been discussed in [7]. Such a computation for Kaluza–Klein quantum numbers of brane modes turn out to be more complicated, and we disregard it.

Finally we note that in the $\mathrm{SL}(2,\mathbb{Z}+\mathrm{i}\mathbb{Z})/\{\pm\mathrm{Id}\}$ orbifold the instability may be absent. To demonstrate that for three-orbifold we can use the arguments given for string models in [7]. The instability may originate from the gravitational interaction of plane waves. The continuum part of the spectrum may lead to wave interactions, but it is severely restricted by $\mathrm{SL}(2,\mathbb{Z}+\mathrm{i}\mathbb{Z})/\{\pm\mathrm{Id}\}$ symmetry, and the argument of [26] of instability does not seem to directly apply to our case. In the discrete part of the spectrum the states have a finite motion, and the corresponding wave functions are regular.

5 Cosets $\Gamma\backslash G/K$ and Killing spinors

In the previous sections we have considered real hyperbolic space forms. The hyperbolic spaces $\mathbb{H}^N$ have Killing spinors transforming in the spinorial representation of $\mathrm{SO}_1(N-1,1)$ [27] (see also [6,28,29]). Thus the simplest membrane model with trivial Γ allows for supersymmetry. In general, the following results hold, a proposition by Friedrich [30].

A Riemannian spin manifold (M^N, g) admitting a Killing spinor $\psi\neq 0$ with Killing number $\mu\neq 0$ is locally irreducible. we can proof this as follows.

Let the locally Riemannian product have the form $M^N = M^K\times M^{N-K}$. Let $\mathcal{X},\mathcal{Y}$ be vectors tangent to M^K and M^{N-K} respectively, and, therefore, the curvature tensor of the Riemannian manifold (M^N, g) be trivial. Since ψ is a Killing spinor the following hold:

$$\begin{aligned}\nabla_{\mathcal{X}}\psi &= \mu\mathcal{X}\cdot\psi, \\ 4\mu^2 &= [N(N-1)]^{-1}R \ \text{ at each point of a} \\ &\quad \text{connected Riemannian spin manifold } \left(M^N, g\right),\end{aligned} \tag{5.1}$$

where R is the scalar curvature. Because of (5.1) we have

$$\begin{aligned}&\nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\psi = \mu(\nabla_{\mathcal{X}}\mathcal{Y})\cdot\psi + \mu^2\mathcal{Y}\cdot\mathcal{X}\cdot\psi \Longrightarrow \\ &(\nabla_{\mathcal{X}}\nabla_{\mathcal{Y}} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}} - \nabla_{[\mathcal{X},\mathcal{Y}]})\psi \\ &\quad = \mu^2(\mathcal{Y}\cdot\mathcal{X} - \mathcal{X}\cdot\mathcal{Y})\psi.\end{aligned} \tag{5.2}$$

The curvature tensor $R(\mathcal{X},\mathcal{Y})$ in the spinor bundle $\mathfrak{S}$ is related to the curvature tensor of the Riemannian manifold (M^N, g): $R(\mathcal{X},\mathcal{Y}) = (1/4)\sum_{j=1}^N e_j R(\mathcal{X},\mathcal{Y})e_j\cdot\psi$, where $\{e_j\}_{j=1}^N$ is an orthogonal basis in the manifold. Therefore (5.2) can also be written

$$\sum_{j=1}^N e_j R(\mathcal{X},\mathcal{Y})e_j\psi + [N(N-1)]^{-1}R(\mathcal{X}\mathcal{Y}-\mathcal{Y}\mathcal{X})\psi = 0. \tag{5.3}$$

From (5.3) we get $R \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \psi = 0$, and moreover $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal vectors. Since $\mu \neq 0$ ($R \neq 0$) it follows that $\psi = 0$; hence there a contradiction.

We have also the following statement due to Friedrich [30].

Let (M^N, g) be a connected Riemannian spin manifold and let ψ be a non-trivial Killing spinor with Killing number $\mu \neq 0$. Then (M^N, g) is an Einstein space. The proof easily follows from the proposition with (2.5) and (2.6); indeed (M^N, g) is an Einstein space of scalar curvature given by (5.1).

There are no normalizable modes for any field configurations in hyperbolic spaces. Spaces with finite volume for a fundamental domain can be obtained by forming the coset spaces with topology $\Gamma \backslash \mathbb{H}^N$ where Γ is a discrete subgroup of the isometry group. Let us comment on the supersymmetry of these spaces following the lines of [6,7]. For non-trivial Γ and finite volume space $\Gamma \backslash \mathbb{H}^N$ it has been shown [6] that for even N supersymmetries are always broken by the identifications. Indeed, the isometry group of $\mathbb{H}^N$ is $\mathrm{SO}_1(N, 1)$ and Γ is in general a subgroup of $\mathrm{SO}_1(N, 1)$, which may or may not have fixed points. Killing spinors are in the spinorial representation of $\mathrm{SO}_1(N-1, 1)$, and if Γ is a subgroup of $\mathrm{SO}_1(N-1, 1)$ but it is not a subgroup of $\mathrm{SO}_1(N-3, 1)$, then there are no surviving Killing spinors. The latter exist if $\Gamma \in \mathrm{SO}_1(N-3, 1)$, but in this case $\Gamma \backslash \mathbb{H}^N$ will still be of infinite volume. Therefore, for even N there are no finite volume cosets $\Gamma \backslash \mathbb{H}^N$ with unbroken supersymmetries. On the other hand, for odd N this analysis does not exclude that an appropriate choice of Γ could give a supersymmetric model with finite volume hyperbolic space. For odd N there are two Killing spinors on $\mathbb{H}^N$ in the spinorial representation of $\mathrm{SO}_1(N-1, 1)$. These spinors are also Weyl spinors of the isometry group $\mathrm{SO}_1(N, 1)$, so they form an irreducible Dirac spinor of $\mathrm{SO}_1(N, 1)$. All supersymmetries are broken if Γ is not a subgroup $\mathrm{SO}_1(N-1, 1)$. If Γ is a subgroup $\mathrm{SO}_1(N-1, 1)$, then half of the supersymmetries survive. A question of interest is whether supersymmetry survives under the orbifolding by the discrete group Γ. Perhaps there are more solutions involving real hyperbolic spaces, where some supersymmetries are unbroken. However the analysis of that problem is complicated and we leave it for another occasion.

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